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Duality theorems for a new class of multitime multiobjective variational problems

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AIM(1):The value of the objective function of the primal cannot exceed the value of dual.

(2) To study the connectivity between the values of the objectives function of the primal and dual programming in direct and converse duality programming.

Abstract In this work, we consider a new class of multitime multiobjective variational problems of minimizing a vector of functionals of curvilinear integral type. Based on the normal efficiency conditions for multitime multiobjective variational problems, we study duals of Mond-Weir type, generalized Mond-Weir-Zalmai type and under some assumptions of *(ρ, b)*-quasiinvexity, duality theorems are stated. We give weak duality theorems, proving that the value of the objective function of the primal cannot exceed the value of the dual. Moreover, we study the connection between values of the objective functions of the primal and dual programs, in direct and converse duality theorems. While the results in §1 and §2 are introductory in nature, to the best of our knowledge, the results in §3 are new and they have not been reported in literature.

Keywords Multitime multiobjective problem · Efficient solution · Quasiinvexity · Duality

Mathematics Subject Classification (2000) 65K10 · 90C29 26B25 · 26B25

Introduction and statement of the problems

Necessity conditions of optimality for scalar variational problems were introduced and studied by Valentine [27]. The duality of scalar variational problems involving convex and gen- eralized convex functions was further developed by Mond and Hanson [13], Mond, Chandra and Husain [14], Mond and Husain [15], Preda [21]. Mititelu developed a duality for the mult- itime scalar control problem with mixed constraints, using the invexity notion, [9]. Under various types of generalized convex functions, Mukherjee and Purnachandra Rao developed mixed type duality results [16], Preda and Gramatovici proved sufficient optimality conditions for multiobjective variational problems [23], while Mititelu and collaborators [7,8,10,11], Preda [22], Zalmai [28], established several weak efficiency conditions and developed dif- ferent types of dualities for multiobjective fractional variational problems. Jagannathan [5] studied several optimality and duality results for nonlinear fractional programming. Kim and Kim [6] used the efficiency property of nondifferentiable multiobjective variational problems in duality theory under generalized convexity assumptions. Despite of all these important advances, our multitime multiobjective problem -imposed by practical reasons- had not been studied so far.

Inspired and motivated by the ongoing research in this area, we introduce and study a new class of multitime multiobjective variational problems of minimizing a vector of functionals of curvilinear integral type. Using essentially the techniques of Ariana Pitea [18] and his research collaborators, our results establish certain new conditions of Mond-Weir-Zalmai duality type for multitime multiobjective variational problems based on the notion of *(ρ, b)*-quasiinvexity.

To introduce our study problem, we need the following background, which is necessary for the completeness of the exposition. For more details, we address the reader to [19] and [20].

Let (T, h) and (M, g) be Riemannian manifolds of dimensions p and n, respectively. Denote by $t=\left(t^{\alpha}\right), \alpha=\overline{1,p}$, and $x=\left(x^{i}\right), i=\overline{1,n}$, the local coordinates on T and M, respectively. Denote by J^1 (T, M) the first order jet bundle associated to T and M.

Using the product order relation on $\mathbf{R}^p$, [10], the hyperparallelepiped Ω_{t_0,t_1}, $\mathbf{R}^p$. with diagonal opposite points $t_0=(t_0^1,\ldots,t_0^p)$ and $t_1=(t_1^1,\ldots,t_1^p)$, can be written as being the interval $[t_0, t_1]$. Suppose γ_{t_0,t_1} is a piecewise C^1 -class curve joining the points t_0 and t_1.

The closed Lagrange I-forms densities of C^{∞}-class.

$$f_{\alpha}=\left(f_{\alpha}^{\ell}\right): J^1\left(T,M\right)\to \mathbf{R}^r,\ \ell=\overline{1,r},\ \ \alpha=\overline{1,p}.$$

determine the following path independent curvilinear functionals (actions)

$$F^{\ell}\left(x(\cdot)y(\cdot)\right)=\int_{\gamma_{t_0,t_1}} f_{\alpha}^{\ell}\left(t,x\left(t\right),x_{\gamma}(t);t,y(t),y_{\gamma}(t)\right)dt^{\alpha},$$

where $x_{\gamma}(t)=\dfrac{\partial x}{\partial t^{\gamma}}(t), y_{\gamma}(t)=\dfrac{\partial y}{\partial t^{\gamma}}(t), \gamma=\overline{1,p}$ are partial velocities.

Important note: To simplify the notations, in our subsequent theory, we shall set

$$\pi_x(t)=(t,x(t),x_{\gamma}(t)),\ \ \pi_{x^0}(t)=(t,x^o(t),x_{\gamma}^o(t)),\ \pi_y(t)=(t,y(t),y_{\gamma}(t)).$$

$$\pi_{y^0}(t)=(t,y^0(t),y_{\gamma}^0(t)),\ \pi_z(t)=(t,z(t),z_{\gamma}(t))$$

The closeness conditions (complete integrability conditions) are:

$$D_{\beta}f_{\alpha}^{\ell}=D_{\alpha}f_{\beta}^{\ell},\ \alpha,\beta=\overline{1,p},\ \alpha\neq\beta,\ \ell=\overline{1,r},$$ where D_{β} is the total derivative.

Accept that the Lagrange matrix densities

$$g=\left(g_a^b\right): J^1\left(T,M\right)\to \mathbf{R}^{qs},\ a=\overline{1,s},\ b=\overline{1,q},\ \ q<n,$$

$$h=\left(h_a^b\right): J^1\left(T,M\right)\to \mathbf{R}^{qs},\ a=\overline{1,s},\ b=\overline{1,q},\ \ q<n,$$

of C^{∞} – class define the partial differential inequation (PDI) (of evolution)

$$g\left(\pi_x(t)\right) \leqq 0,\ \ t \in \Omega_{\tau_0,t_1},$$

$$g\left(\pi_y(t)\right) \leqq 0,\ \ t \in \Omega_{\tau_0,t_1},$$

and the partial differential equations (PDE) (of evolution)

$$h\left(\pi_x(t)\right) = 0,\ \ t \in \Omega_{\tau_0,t_1}.$$

$$h\left(\pi_y(t)\right) = 0,\ \ t \in \Omega_{\tau_0,t_1}.$$

On the set $C^{\infty}(\Omega_{\tau_0,t_1}, M)$ of all functions $x, y : \Omega_{\tau_0,t_1} \to M$ of C^{∞}-class, we set the norm

$$\|x\| = \|x\|_{\infty} + \sum_{\alpha=1}^{p} \| x_{\alpha} \|_{\infty}.$$

$$\|y\| = \|y\|_{\infty} + \sum_{\alpha=1}^{p} \| y_{\alpha} \|_{\infty}.$$

Let

$$F\left(\Omega_{t_0,t_1}\right) = \left\{x, y \in C^{\infty}\left(\Omega_{t_0,t_1}, M\right) \mid x\left(t_0\right) = x_0, y(t_0) = y_0, x(t_1) = x_1, y(t_1) = y_1, \text{ or } x(t)|_{\partial\Omega_{t_0,t_1}} = \chi = \text{given}\right.$$
$$\left. g(\pi_x(t); \pi_y(t)) \leqq 0, h(\pi_x(t); \pi_y(t)) = 0, t \in \Omega_{t_0,t_1}\right\}$$

be the set of all feasible solutions of problem (MP), introduced right now.

Denote by $F(x(\cdot), y(\cdot)) = \left(F^1(x(\cdot), y(\cdot)).....F^r\left(x(\cdot), y(\cdot)\right)\right).$

The aim of this work is to introduce and study the variational problem of minimizing a vector of functional of curvilinear integral type.

$$(MP)\begin{cases} \text{Min } F(x, y) \\ \text{ subject to } x(\cdot), y(\cdot) \in F(\Omega_{t_0,t_1}). \end{cases}$$

According to Chinchuluun and Pardalos [2], most of the optimization problems arising in practice have several objectives which have to be optimized simultaneously. This kind of problems, of considerable interest, includes various branches of mathematical sciences, engineering design, portfolio selection, game theory, decision problems in management science, web access problems, query optimization in databases etc. For descriptions of the web access problem, the portfolio selection problem and capital budgeting problem, see [2,3], and some references therein.

Our study is motivated by its deep application especially in Mechanical Engineering, where curvilinear integral objectives are extensively used due to their physical meaning as mechanical work. These objectives play an essential role in mathematical modeling of certain processes in relation with Robotics, Tribology, Engines etc. In mathematical terms, in (MP) we are given a number of *r* sources producing mechanical work, which have to be minimized on a set of limited resources, namely $F(\Omega_{t_0,t_1})$.

In their article [19], Ariana Pitea, C. Udris¸te and S¸ t. Mititelu established necessary efficiency conditions for problem (MP). The present paper is thought to be natural continuation of the research in [19], studying the problem of duality on the first order jet bundle $J^1(T, M)$. The choice of this framework was imposed by physical considerations regarding the mechanical work.

To state and prove our efficiency conditions and Mond-Weir-Zalmai duality for problem (MP), it is necessary to recall some definitions and auxiliary results which will be needed later in our discussion.

Definition 1: A feasible solution $x^0(\cdot), y^0(\cdot) \in F(\Omega_{t_0,t_1})$ is called an efficient solution of problem (MP) if there is no $x(\cdot), y(\cdot) \in F(\Omega_{t_0,t_1})$, $x(\cdot) \neq x^0(\cdot)$ and $y(\cdot) \neq y^0(\cdot)$ such that

$$F(x(\cdot), y(\cdot)) \leq F(x^0(\cdot), y^0(\cdot)).$$

Let ρ be a real number and $b: C^\infty(\Omega_{t_0,t_1}, M) \times C^\infty(\Omega_{t_0,t_1}, M) \to [0,\infty)$ a functional. To any closed I-form $a = (a_\alpha)$ we associate the path independent curvilinear functional

$$A(x(\cdot), y(\cdot)) = \int_{\gamma_{t_0,t_1}} a_\alpha\left(\pi_x(t), \pi_y(t)\right) dt^\alpha$$

The following definition of quasiinvexity [20], will be proved to be of paramount importance, helping us to state our main results included in this work.

Definition: 2 The functional A is called [strictly] (ρ, b)-quasiinvex at $(x^0(\cdot), y^0(\cdot))$ if there is a vector function $\eta: J^1\left(\Omega_{t_0,t_1}, M\right) \times J^1\left(\Omega_{t_0,t_1}, M\right) \to \mathbf{R}^n$, with $\eta\left(\pi_{x^0}(t), \pi_{y^0}(t)\right) = 0$, and the function $\theta: C^\infty\left(\Omega_{t_0,t_1}, M\right) \times C^\infty\left(\Omega_{t_0,t_1}, M\right) \to \mathbf{R}^n$, such that for any $(x(\cdot), y(\cdot))[x(\cdot), y(\cdot) \neq x^0(\cdot), y^0(\cdot)]$ the following inmplication holds.

$$(A(x(\cdot), y(\cdot)) \leq A(x^0), y^0(\cdot)))$$

$$\Rightarrow \Big(b(x(\cdot), x^0(\cdot); y(\cdot), y^0(\cdot))$$

$$\int_{\gamma_{t_0,t_1}} \Big[< \eta(\pi_x(t), \pi_{x^0}(t); \pi_y(t), \pi_{y^0}(t)), \frac{\partial a_\alpha}{\partial x}(\pi_{x^0}(t); \pi_{y^0}(t)) >$$

$$+ < D_\gamma \eta\left(\pi_x(t), \pi_{x^0}(t); \pi_y(t), \pi_{y^0}(t)\right), \frac{\partial a_\alpha}{\partial x_\gamma}(\pi_{x^0}(t); \pi_{y^0}(t)) > \Big] dt^\alpha$$

$$\leq -\rho b(x(\cdot), x^0(\cdot); y(\cdot), y^0(\cdot)) \| \theta(x(\cdot), x^0(\cdot); y(\cdot), y^0(\cdot)) \|^2 \Big)$$

In the following example, we consider two functional of curvilinear integral type which are (ρ, b)-quasiinvex, in each case providing the function η.

Example1: Let $a: [0,1] \times C^\infty\left([0,1]\right) \to \mathbf{R}, x(\cdot), y(\cdot) = (x^1(\cdot), x^2(\cdot); y^1(\cdot), y^2(\cdot))$. As it can be verified, the functional

$$A(x(\cdot), y(\cdot)) = \int_0^1 a\left(t, x(t); t, y(t)\right) dt$$

is $(\rho, 1)$-quasiinvex, for $\rho \leq 0$ and any θ, at the point (x^0, y^0) with respect to

$$\eta(\pi_x(t),\pi_{x^0}(t);\pi_y(t),\pi_{y^0}(t)) = (A(x(\cdot),y(\cdot)) - A(x^0(\cdot),y^0(\cdot))$$
$$\left(\frac{\partial a}{\partial x^1}\left(t,x^0(t);t,y^0(t)\right),\frac{\partial a}{\partial x^2}\left(t,x^0(t);t,y^0(t)\right)\right).$$

In a similar manner, the functional

$$A(x(\cdot),y(\cdot)) = \int_0^1 a\left(t,\overline{x}(t);t,\overline{y}(t)\right)dt$$

is $(\rho,1)$-quasiinvex, for $\rho \le 0$ and any θ, at the point (x^0,y^0) with respect to $\eta(\pi_x(t),\pi_{x^0}(t);\pi_y(t),\pi_{y^0}(t)) = -(A(x(\cdot),y(\cdot)) - A(x^0(\cdot),y^0(\cdot))$

$$\times\left(D\frac{\partial a}{\partial \dot{x}^1}(t,\overline{x}^0(t);t,\overline{y}^0(t)),D\frac{\partial a}{\partial \dot{x}^2}(t,\overline{x}^0(t);t,\overline{y}^0(t))\right).$$

where by *D*, we denoted the total derivative operator.

It can be seen that these examples can be easily extended to *n*-dimensional vector valued functions and, also, for the multitime case (using normal coordinates).

We underline that there is no general method for the computation of η, for given classes of multitime multiobjective variational problems. However, the notion of quasiinvexity is extensively used, in appropriate forms with general η, in recent works for studies of some multiobjective programming problems. In [1] by Antczak, several optimality results are obtained for a modified ratio objective problem; in [4] by Husain and Jabeen, mixed type duality the- orems are stated for problems containing support functions; in [24] by Puglisi, the subject of generalized convexity and invexity is studied; in [17] by Nahak and Mohapatra, nonsmooth invexity is used to study some multiobjective programming problems, while in [12] and [25], Mititelu and Stancu-Minasian introduce a study of some multiobjective fractional variational problems via adequate assumptions on quasiinvexity. That is why, the present work proposes the study for our class of problems, employing a general η.

In order to harmonize the new results exposed in Sect. 3 of this work with the assembly of our current research, we consider suitable to recall the statements from Sect. 2.

Multitime multiobjective Mond-Weir type duality for (MP)

Consider a function $z(\cdot) \in X$ *and associate to problem (MP) the multitime multiobjective variational problem*

$$
(MD)\begin{cases}
\max\limits_{z(\cdot)}\left(F^1(z(\cdot)),\ldots F^r(z(\cdot))\right) \\
\text{Subject to} \\
\Lambda_\ell \dfrac{\partial f_\alpha^\ell}{\partial z}(\pi_z(t)) + <\mu_\alpha(t), \dfrac{\partial g}{\partial z}(\pi_z(t))> + <\nu_\alpha(t), \dfrac{\partial h}{\partial z}(\pi_z(t))> \\
-D_\gamma\left(\Lambda_\ell \dfrac{\partial f_\alpha^\ell}{\partial z_\gamma}(\pi_z(t)) + <\mu_\alpha(t), \dfrac{\partial g}{\partial z_\gamma}(\pi_z(t))> + <\nu_\alpha(t), \dfrac{\partial h}{\partial z_\gamma}(\pi_z(t))>\right) \\
= 0, \quad \alpha = \overline{1,p}, \quad t \in \Omega_{t_0,t_1} \\
<\mu_{\alpha a}(t), g^a(\pi_z(t))> + <\nu_\alpha(t), h^a(\pi_z(t))> \geq 0 \\
\qquad\qquad a = \overline{1,q}, \alpha = \overline{1,p}, \ t \in \Omega_{t_0,t_1} \\
\Lambda \geq 0.
\end{cases}
$$

taking into account that the function z(t) has to satisfy the boundary conditions z(t₀)= $(x_0, y_0), z(t_1) = (x_1 y_1)$, *or* $z(t)|_{\partial\Omega_{t_0,\tau_1}} = \chi =$ *given, the partial differential inequations of evolution, and the partial differential equations of evolution.*

Denote by $\pi(x^0(\cdot), y^0(\cdot))$ the minimizing functional vector of problem (MP) at the point $x^0(\cdot), y^0(\cdot) \in F(\Omega_{t_0,t_1})$ and by $\delta(z(\cdot), z_\gamma(\cdot)\Lambda, \mu(\cdot), \nu(\cdot), \tau(\cdot))$ the maximizing functional vector of dual problem (MD) at $\left(z(\cdot), z_\gamma(\cdot)\Lambda, \mu(\cdot), \nu(\cdot), \tau(\cdot)\right)$ in Δ, where Δ is the domain of the problem (MD). We introduce three duality results.

Theorem 1 *(WEAK DUALITY) Let* $x^0(\cdot), y^0(\cdot)$ *be a feasible solution of problem (MP) and* $\left(z(\cdot), z_\gamma(\cdot)\Lambda, \mu(\cdot), \nu(\cdot), \tau(\cdot)\right)$ *be a feasible solution of problem (MD). Assume that the following conditions are fulfilled:*

(a) For any $\ell = \overline{1,r}$, *the functional* $F^\ell(x(\cdot), y(\cdot))$ *is* $(\rho^{r\ell}, b)$*-quasiinvex at the point* $z(\cdot)$ *with respect to* η *and* θ;

(b) for each $a = \overline{1,q}$ *, the functional*

$$\int_{\gamma_{t_0,t_1}} \left[<\mu_{\alpha a}(t), g^a(\pi_x(t);\pi_y(t))> + <\nu_{\alpha a}(t), h^a(\pi_x(t);\pi_y(t))>\right] dt^\alpha$$

is (ρ_a'', b)*-quasiinvex at* $z(\cdot)$ *with respect to* η *and* θ;

(c) at least one of the functional of (a), (b) is strictly quasiinvex;

(d) $\rho'^\ell \Lambda_\ell + \sum_{a=1}^{q} \rho_a'' \geq 0$

Then, the inequality $\pi(x^0(\cdot), y^0(\cdot)) \le \delta(z(\cdot), z_\gamma(\cdot), \Lambda, \mu(\cdot), \nu(\cdot), \tau(\cdot))$ *is false.*

Proof The proof uses techniques similar to those in [20].

In a previous paper [19], we proved that if $x^0(\cdot), y^0(\cdot)$ is an efficient solution of problem (MP), there is a vector Λ in $\mathbf{R}^r$ and the smooth functions $\mu^0 : \Omega_{t_0,t_1} \to \mathbf{R}^{qsp}, \nu^0 : \Omega_{t_0,t_1} \to \mathbf{R}^{qsp}$, such that we have

$$\Lambda_j \frac{\partial f^J_\alpha}{\partial x}\left(\pi_{x^0}(t); \pi_{y^0}(t)\right) + < \mu^0_\alpha(t), \frac{\partial g}{\partial x}\left(\pi_{x^0}(t); \pi_{y^0}(t)\right) > + < \nu^0_\alpha(t), \frac{\partial h}{\partial x}\left(\pi_{x^0}(t); \pi_{y^0}(t)\right) >$$

$$-D_\gamma\left(\Lambda_j \frac{\partial f^j_\alpha}{\partial x_\gamma}(\pi_{x^0}(t); \pi_{y^0}(t)) + < \mu^0_\alpha(t), \frac{\partial g}{\partial x_\gamma}(\pi_{x^0}(t); \pi_{y^0}(t)) > \right)$$

$$= 0,\ t \in \Omega_{t_0,t_1}, \alpha = \overline{1,p} \text{ (Euler-Lagrange PDEs)}$$

$$< \mu^0_\alpha(t), g(\pi_{x^0}(t); \pi_{y^0}(t)) >= 0, t \in \Omega_{t_0,t_1}, \alpha = \overline{1,p},$$

$$\mu^0_\alpha(t) \geqq 0,\ t \in \Omega_{t_0,t_1}, \alpha = \overline{1,p}.$$

If $\Lambda \ge 0$, then $x^0(\cdot), y^0(\cdot)$ from conditions (2.1) is called normal efficient solution.

Let $x^0(\cdot), y^0(\cdot)$ be a normal efficient solution of primal (MP), the scalar Λ in $\mathbf{R}^r$ and the smooth functions $\mu^0 : \Omega_{t_0,t_1} \to \mathbf{R}^{qsp}, \nu^0 : \Omega_{t_0,t_1} \to \mathbf{R}^{qsp}$, given above.

We are in position to state a direct duality result. This is given by

Theorem 2 (Direct Duality) Suppose that the hypotheses of Theorem 1 are satisfied. Then $(x^0(\cdot), x^0_\gamma(\cdot); y^0(\cdot), y^0_\gamma(\cdot)\Lambda, \mu^0(\cdot), \nu^0(\cdot)\tau^0(\cdot)$ is an efficient solution of dual program (MD) and $\pi(x^0(\cdot), y^0(\cdot)) = \ \delta(x^0(\cdot), x^0_\gamma(\cdot); y^0(\cdot), y^0_\gamma(\cdot)\Lambda, \mu^0(\cdot), \nu^0(\cdot)\tau^0(\cdot))$.

Since the notion of efficient solution of problem (MD) is similar to those given in Definition 1, we shall present now a result concerning the converse duality. By changing some of the hypotheses.

Theorem 3 (Converse Duality) Let $(x^0(\cdot), x^0_\gamma(\cdot); y^0(\cdot), y^0_\gamma(\cdot)), \Lambda, \mu^0(\cdot), \nu^0(\cdot), \tau^0(\cdot))$ be an efficient solution of dual problem (MD) and suppose that the following conditions are fulfilled:

(a) $\left(x^0(\cdot), y^0(\cdot)\right)$ is a feasible solution of primal problem (MP);

(b) for each $\ell = \overline{1,r}$, the functional $F^\ell(x(\cdot), y(\cdot))$ is (ρ'^ℓ, b)-quasiinvex at the point $\left(x^0(\cdot), y^0(\cdot)\right)$ with respect to η and θ;

(c) for each $a = \overline{1,q}$, the functional

$$\int_{\gamma_{t_0,t_1}} \left[< \mu_{\alpha a}(t), t^\alpha\left(\pi_x(t); \pi_y(t)\right) > + < \nu_{\alpha a}(t), h^\alpha(\pi_x(t); \pi_y(t)) > \right] dt^\alpha$$

is (ρ''_a, b)-quasiinvex at the point $x^0(\cdot), y^0(\cdot)$ with respect to η and θ;

(d) at lest one of the functionals of (b), (c) is strictly quasiinvex with respect to η and θ, respectively

(e) $\rho'^\ell \Lambda_\ell + \sum_{a=1}^{q} \rho''_a \ge 0.$

Then $x^0(\cdot), y^0(\cdot)$ is an efficient solution of primal (MP) and $\pi(x^0(\cdot), y^0(\cdot)) = \delta(\{x^0(\cdot), x^0_\gamma(\cdot); y^0(\cdot), y^0_\gamma(\cdot)\}, \Lambda, \mu^0(\cdot), \nu^0(\cdot), \tau^0(\cdot))$

3. Mond-Weir-Zalmai duals for (MP)

In this section, a more general context is needed. That is why we consider the Lagrange matrix densities

$$g=\left(g_a^b\right):J^1\left(T,M\right)\to\mathbf{R}^{\bar{q}s},\ a=\overline{1,s},\ b=\overline{1,\bar{q}},\ \bar{q}<n,$$

$$h=\left(h_a^b\right):J^1\left(T,M\right)\to\mathbf{R}^{\bar{q}s},\ a=\overline{1,s},\ b=\overline{1,\bar{q}},\ \bar{q}<n,$$

of C^∞-class define the partial differential inequations (PDI) (of evolution)

$$g\left(\pi_x(t);\pi_y(t)\right)\leqq 0,\ t\in\Omega_{t_0,t_1},$$

and the partial differential equations (PDE) (of evaluation)

$$h\left(\pi_x(t);\pi_y(t)\right)=0,\ t\in\Omega_{t_0,t_1},$$

In order to use the idea of "grouping the resources", consider $[p_0, p_1, \ldots, p_q]$ and $\{Q_0, Q_1, \ldots Q_q\}$ partitions of the sets $\{1,\ldots.\bar{p}\}$ and $\{1,\ldots.\tilde{q}\}$ respectively.

For each $\ell=\overline{1,r}$ and $\alpha=\overline{1,p}$, we denote

$$\bar{f}_\alpha^\ell(\pi_z(t))=f_\alpha^\ell(\pi_z(t))+<\mu_{\alpha P_0}(t),g^{P_0}(\pi_z(t))>+<\nu_{\alpha Q_0}(t),h^{Q_0}(\pi_z(t))>,$$

and $\bar{F}^\ell(z(\cdot))=\int\limits_{\gamma_{t_0,t_1}}\bar{f}_\alpha^\ell(\pi_z(t))dt^\alpha$

Consider a function $z(\cdot)\in X$ and associate to (MP) the multiobjective variational problem

$$(MZD)\begin{cases}\max\limits_{z(\cdot)}(\bar{F}^1(z(\cdot)),\ldots,\bar{F}^r(z(\cdot))\\ \text{Subject to}\\ \Lambda_\ell\dfrac{\partial f_\alpha^\ell}{\partial z}\left(\pi_z(t))+<\mu_\alpha(t),\dfrac{\partial g}{\partial z}\left(\pi_z(t)\right)>+<\nu_\alpha(t),\dfrac{\partial h}{\partial z}(\pi_z(t))>\right)\\ -D_\gamma\left(\Lambda_\ell\dfrac{\partial f_\alpha^\ell}{\partial z_\gamma}(\pi_z(t))+<\mu_\alpha(t),\dfrac{\partial g}{\partial z_\gamma}(\pi_z(t))>+<\nu_\alpha(t),\dfrac{\partial h}{\partial z_\gamma}(\pi_z(t))>\right)\\ =0,\ \alpha=\overline{1,p},\ t\in\Omega_{t_0,t_1}\\ <\mu_{\alpha P_k}(t),g^{P_k}(\pi_z(t))>+<\nu_{\alpha Q_k}(t),h^{Q_k}(\pi_z(t))>\geq 0,\\ \qquad K=\overline{1,q},\alpha=\overline{1,p},t\in\Omega_{t_0,t_1}\\ \Lambda\geq 0,\end{cases}$$

taking into account that the function y(t) has to satisfy the boundary conditions $z(t_0)=(x_0,y_0)$ $z(t_1)=(x_1,y_1)$, or $z(t)|_{\partial\Omega_{t_0,t_1}}=\chi=$ given, the partial differential inequations of evolution, and the partial differential equations of evolution.

$\pi(x^0(\cdot),y^0(\cdot))$ is the value of the objective function of problem (MP) at $x^0(\cdot),y^0(\cdot)\in F(\Omega_{t_0,t_1})$ and $\delta(z(\cdot),z_\gamma(\cdot),\Lambda,\mu(\cdot),\nu(\cdot),\tau(\cdot))$ is the maximizing functional vector of dual problem (MZD) at $(z(\cdot), z_{\gamma}(\cdot), \Lambda, \mu(\cdot), \nu(\cdot), \tau(\cdot)) \in \Delta$, where Δ is the domain of problem(MZD).

We introduce three duality results in the sense of Mond-Weir-Zalmai.

Theorem 4 (Weak Duality) Let $x^0(\cdot), y^0(\cdot)$ be a feasible solution of problem (MP) and $\delta(z(\cdot), z_{\gamma}(\cdot), \Lambda, \mu(\cdot), \nu(\cdot), \tau(\cdot))$ be a feasible point of problem (MZD). Assume that the following conditions are satisfied.

(a) $<\mu_{p_0}(t), g^{P_0}(\pi_z(t))> + <\nu_{Q_0}(t), h^{Q_0}(\pi_z(t))> \leq 0,$

(b) For each $\ell = \overline{1,r}$, $\Lambda_{\ell} F^{\ell}(x(\cdot), y(\cdot))(\rho'^{\ell}, b)$-quasiinvex at the point $z(\cdot)$ with respect to η and θ.

(c) The functional $\int_{\gamma_{t_0,t_1}} \left[<\mu_{\alpha} P_k(t), g^{P_k}(\pi_x(t);\pi_y(t))> + <\nu_{\alpha Q_K}(\pi_x(t);\pi_y(t))>\right] dt^{\alpha}$

is (ρ'^{ℓ}, b)-quasiinvex at $z(\cdot)$ with respect to η and θ, for each $k = \overline{1,q}$;

(d) At least one of the functions of (b), (c) is strictly quasiinvex;

(e) $\rho'^{\ell} \Lambda_{\ell} + \sum_{k=1}^{q} \rho''_k \geq 0.$

Then, the inequality $\pi\left(x^0(\cdot), y^0(\cdot)\right) \leq \overline{\delta}(z(\cdot), z_{\gamma}(\cdot)\Lambda, \mu(\cdot), \nu(\cdot), \tau(\cdot)$ is false.

Proof: By reduction ad absurdum, suppose $F^{\ell}(x^0(\cdot), y^0(\cdot)) \leq \overline{F}^{\ell}(z(\cdot)),\ \ell = \overline{1,r}$

From these inequalities, it follows

$$F^{\ell}(x^0(\cdot), y^0(\cdot)) \leq F^{\ell}(z(\cdot))$$

$$+ \int_{\gamma_{t_0,t_1}} \left[<\mu_{\alpha} P_0(t), g^{P_0}(\pi_z(t))> + <\nu_{\alpha} Q_0(t), h^{Q_0}(\pi_z(t))>\right] dt^{\alpha}$$

And taking into account the hypothesis (a) , we get

$$F^{\ell}(x^0(\cdot), y^0(\cdot)) \leq F^{\ell}(z(\cdot)),\ \forall \ell = \overline{1,r} \tag{3.1}$$

We multiply by Λ_{ℓ} and we use hypothesis (b), Making the sum from $\ell = 1$ to $\ell = r$, it follows

$$\left(\Lambda_{\ell}\left(F^{\ell}(x^0(\cdot), y^0(\cdot)) - F^{\ell}(z(\cdot))\right) \leq 0\right)$$

$$\Rightarrow \left(b\left(x^0(\cdot), y^0(\cdot), z(\cdot)\right) \int_{\gamma_{t_0,t_1}} \left[<\eta\left(\pi_{x^0}(t), \pi_{y^0}(t), \pi_z(t)\right), \Lambda_{\ell} \frac{\partial f_{\alpha}^{\ell}}{\partial z}\left(\pi_z(t)\right)> \right.\right.$$

$$\left. + <D_{\gamma}\eta(\pi_{x^0}(t), \pi_{y^0}(t), \pi_z(t)), \Lambda_{\ell} \frac{\partial f_{\alpha}^{\ell}}{\partial z_{\gamma}}(\pi_z(t))> \right] dt^{\alpha}$$

$$\left. \leq -b\left(x^0(\cdot), y^0(\cdot), z(\cdot)\right) \left\| \theta\left(x^0(\cdot), y^0(\cdot), z(\cdot)\right) \right\|^2 \rho'' \Lambda_{\ell} \right) \tag{3.2}$$

Hypothesis c), regarding the (ρ''_k, b) quasiinvexity property of each functional, implies $(k=\overline{1,q})$

$$\left(\int_{\gamma_{t_0,t_1}} \left[<\mu_\alpha P_0(t), g^{P_0}(\pi_{x^0}(t);\pi_{y^0}(t))> + <\nu_\alpha Q_k(t), h^{Q_k}(\pi_{x^0}(t);\pi_{y^0}(t))> \right] dt^\alpha \right.$$

$$\left. \leq \int_{\gamma_{t_0,t_1}} \left[<\mu_\alpha P_0(t), g^{P_0}(\pi_z(t))> + <\nu_\alpha Q_k(t), h^{Q_k}(\pi_z(t))> \right] dt^\alpha \right)$$

$$\Rightarrow \Big(b(x^0(\cdot), y^0(\cdot), z(\cdot))$$

$$\int_{\gamma_{t_0,t_1}} <\eta(\pi_{x^0}(t), \pi_{y^0}(t), \pi_z(t)), <\mu_0 P_k(t), \frac{\partial g^{P_k}}{\partial z}(\pi_z(t))>$$

$$+ <\nu_{\alpha Q_k}(t), \frac{\partial h^{Q_k}}{\partial z}(\pi_z(t))>> + <D_y\eta(\pi_{x^0}(t), \pi_{y^0}(t), \pi_z(t)).$$

$$<\mu_\alpha P_k(t), \frac{\partial g^{P_k}}{\partial z_\gamma}(\pi_z(t))> + <\nu_{\alpha Q_k}(t), \frac{\partial h^{Q_k}}{\partial z_\gamma}(\pi_z(t))>> dt^\alpha$$

$$\leq -b\big(x^0(\cdot), y^0(\cdot), z(\cdot)) \,\|\, \theta(x^0(\cdot), y^0(\cdot), z(\cdot)) \,\|^2 \, \rho''_K \big). \qquad (3.3)$$

Now, we make the sum of implications (3.2) and (3.3) side by side and from k=1 to k =q. It follows

$$\Big(\Lambda_\ell \big(F^\ell(x^0(\cdot), y^0(\cdot)) - F^\ell(z(\cdot)) \big)$$

$$+ \int_{\gamma_{t_0,t_1}} \left[<\mu_\alpha(t), g(\pi_{x^0}(t);\pi_{y^0}(t))> + <\nu_\alpha(t), h(\pi_{x^0}(t);\pi_{y^0}(t))> \right] dt^\alpha$$

$$\left. - \int_{\gamma_{t_0,t_1}} [<\mu_\alpha(t), g(\pi_z(t))> + <\nu_\alpha(t), h(\pi_z(t))>] dt^\alpha \leq 0 \right)$$

$$\Rightarrow \left(B(x^0(\cdot), y^0(\cdot), z(\cdot) \int_{\gamma_{t_0,t_1}} <\eta(\pi_{x^0}(t); \pi_{y^0}(t); \pi_z(t)), \Lambda_\ell \frac{\partial f^\ell_\alpha}{\partial z} \right.$$

$$(\pi_z(t)) + <\mu_\alpha(t), \frac{\partial g}{\partial z}(\pi_z(t))>$$

$$+ <\nu_\alpha(t), \frac{\partial h}{\partial z}(\pi_z(t))>> + <D_y\eta(\pi_{x^0}(t), \pi_{y^0}(t), \pi_z(t)), \Lambda_\ell \frac{\partial f^\ell_\alpha}{\partial z_\gamma}(\pi_x(t))$$

$$+ <\mu_\alpha(t), \frac{\partial g}{\partial z_\gamma}(\pi_z(t))> + <\nu_\alpha(t), \frac{\partial h}{\partial z_\gamma}(\pi_z(t))>> dt^\alpha$$

$$< -b(x^0(\cdot), y^0(\cdot), z(\cdot) \,\|\, \theta(x^0(\cdot), y^0(\cdot), z(\cdot) \,\|^2 \left(\rho'^\ell \Lambda_\ell + \sum_{k=1}^{q} \rho''_k \right) \Big). \qquad (3.4)$$

Since $b(x^0(\cdot), y^0(\cdot), z(\cdot)) > 0$, we obtain

$$\int_{\gamma_{t_0,t_1}} \left(< \eta(\pi_{x^0}(t), \pi_{y^0}(t), \pi_z(t)), \frac{\partial V_\alpha}{\partial z}(\pi_z(t), \Lambda, \mu(\cdot), \nu(\cdot), \tau(\cdot) > \right.$$

$$\left. + < D_\gamma \eta(\pi_{x^0}(t), \pi_{y^0}(t), \pi_z(t)), \frac{\partial V_\alpha}{\partial z_\gamma}(\pi_z(t)), \Lambda, \mu(\cdot), \nu(\cdot), \tau(\cdot) > \right) dt^\alpha$$

$$< - \| \theta(x^0(\cdot), y^0(\cdot), z(\cdot) \|^2 \left(\rho'^{\ell} \Lambda_\ell + \sum_{k=1}^{q} \rho''_k \right), \tag{3.5}$$

Where

$$V_\alpha(t, z(\cdot), z_\gamma(\cdot), \Lambda, \mu(\cdot), \nu(\cdot), \tau(\cdot) = \Lambda_\ell f_\alpha^\ell(\pi_z(t)) + < \mu_\alpha(z(t)), g(\pi_z(t)) >$$
$$+ < \nu_\alpha(t), h(\pi_z(t)) >,$$

With $t \in \Omega_{t_0,t_1}$, and $\alpha = \overline{1,p}$

The following relation holds

$$< D_\gamma \eta(\pi_{x^0}(t), \pi_{y^0}(t), \pi_z(t)), \frac{\partial V_\alpha}{\partial z_\gamma}(\pi_z(t), \Lambda, \mu(t), \nu(t), \tau(t)) >$$

$$= D_\gamma < \eta(\pi_{x^0}(t), \pi_{y^0}(t), \pi_z(t)), \frac{\partial V_\alpha}{\partial z_\gamma}(\pi_z(t), \Lambda, \mu(t), \nu(t), \tau(t)) > -$$

$$< \eta(\pi_{x^0}(t), \pi_{y^0}(t), \pi_z(t)), D_\gamma \left(\frac{\partial V_\alpha}{\partial z_\gamma} \right) (\pi_z(t), \Lambda, \mu(t), \nu(t), \tau(t)) >. \tag{3.6}$$

By replacing relations (3.6) and by using Euler-Lagrange PDE, relation (3.5) becomes

$$\int_{\gamma_{t_0,t_1}} D_\gamma < \eta(\pi_{x^0}(t), \pi_{y^0}(t), \pi_z(t)), \frac{\partial V_\alpha}{\partial z_\gamma}(\pi_z(t), \Lambda, \mu(t), \nu(t), \tau(t)) > dt^\alpha$$

$$< - \| \theta(x^0(\cdot), y^0(\cdot), z(\cdot)) \|^2 \left(\rho'^{\ell} \Lambda_\ell + \sum_{k=1}^{q} \rho''_k \right). \tag{3.7}$$

For $\alpha, \gamma = \overline{1,p}$,let us denote by

$$Q_\alpha^\gamma(t) =< \eta(\pi_{x^0}(t), \pi_{y^0}(t), \pi_z(t)), \frac{\partial V_\alpha}{\partial yz_\gamma}(\pi_z(t), \Lambda, \mu(t), \nu(t), \tau(t) >,$$

and $I = \int_{\gamma_{t_0,t_1}} D_\gamma Q_\alpha^\gamma(t) dt^\alpha$.

According to [26], §9, a total divergence is equal to a total derivative. Consequently, there exists *Q(t)*, with $Q(t_0) = 0$ and $Q(t_1) = 0$, such that $D_\gamma Q_\alpha^\gamma(t) = D_\alpha Q(t)$ and

$$I = \int_{\gamma_{t_0,t_1}} D_\alpha Q(t) dt^\alpha = Q(t_1) - Q(t_0) = 0.$$

Replacing into inequality (3.7), it follows that

$$0 < - \| \theta(x^0(\cdot), y^0(\cdot), z(\cdot)) \|^2 \left(\rho'^{\ell} \Lambda_\ell + \sum_{k=1}^{q} \rho''_k \right).$$

From hypothesis e), the previous relation becomes 0 < 0, that is false. From relation (3.4), it follows

$$0 \le \Lambda_\ell\left(F^\ell(x^0(\cdot),y^0(\cdot))-F^\ell(z(\cdot))\right)+\int_{\gamma_{t_0,t_1}}\Big[<\mu_\alpha(t),g(\pi_{x^0}(t);\pi_{y^0}(t)>$$

$$+<\nu_\alpha(t),h(\pi_{x^0}(t);\pi_{y^0}(t))>]dt^\alpha-\int_{\gamma_{t_0,t_1}}[<\mu_\alpha(t),g(\pi_z(t))>+<\nu_\alpha(t),h(\pi_z(t))>\Big]dt^\alpha.$$

According to the constraints of problems (MP) and (MZD), the above-mentioned relation becomes $\Lambda_\ell(F^\ell(x^0(\cdot),y^0(\cdot))-F^\ell(z(\cdot))>0$. Hence, there is an index i_0 $F^{i_0}(x^0(\cdot),y^0(\cdot))>F^{i_0}(z(\cdot))$.

We conclude that $\left(F^1(x^0(\cdot);y^0(\cdot)),...,F^r(x^0(\cdot);y^0(\cdot))\right)\not\leq\left(F^1(z(\cdot)...,F^r(z(\cdot))\right)$.

Therefore, the inequality $\pi(x^0(\cdot),y^0(\cdot))\le\overline{\delta}(z(\cdot),z_\gamma(\cdot)\Lambda,\mu(\cdot),\nu(\cdot),\tau(\cdot))$ contradicts relations (3.1) and this completes the proof.

The notion of efficient solution of problem (MZD) is similar to those given in Definition 1. Let $x^0(\cdot)$, $y^0(\cdot)$ be a normal efficient solution of primal (MP), the scalar in $\mathbf{R}^r$ and the smooth functions $\mu^0:\Omega_{t_0,t_1}\to\mathbf{R}^{\bar{q}sp},\nu^0:\Omega_{t_0,t_1}\to\mathbf{R}^{\bar{q}sp}$, from relations (2.1).

Theorem 5 (direct duality) If the hypotheses of Theorem 4 are satisfied, then $(x^0(\cdot),(x^0_\gamma(\cdot),y^0(\cdot),y^0_\gamma(\cdot)\Lambda,\mu^0(\cdot),\nu^0(\cdot),\tau(\cdot))$ is an efficient solution of dual program (MZD) and we have the equality $\pi(x^0(\cdot),y^0(\cdot))=\overline{\delta}(x^0(\cdot),x^0_\gamma(\cdot);y^0(\cdot)y^0_\gamma(\cdot),\Lambda,\mu^0(\cdot),\nu^0(\cdot),\tau^0(\cdot))$.

Proof: According to relation (2,1), Sect, 2, $(x^0(\cdot),x^0_\gamma(\cdot),y^0(\cdot),y^0_\gamma(\cdot),\Lambda,\mu^0(\cdot),\nu^0(\cdot),\tau^0(\cdot))$ is a feasible point of problem (MZD) and we have $\overline{F}^\ell(x^0(\cdot),y^0(\cdot))=F^\ell(x^0(\cdot),y^0(\cdot))$. From theorem 4, Sect. 3, the inequality

$\pi(x^0(\cdot),y^0(\cdot))\le\overline{\delta}(x^0(\cdot),x^0_\gamma(\cdot),y^0(\cdot),y^0_\gamma(\cdot)\ \Lambda,\mu^0(\cdot),\nu^0(\cdot),\tau^0(\cdot))$ is false. Therefore, it follows $\pi(x^0(\cdot),y^0(\cdot))=\overline{\delta}(x^0(\cdot),x^0_\gamma(\cdot),y^0(\cdot),y^0_\gamma(\cdot),\ \Lambda,\mu^0(\cdot),\nu^0(\cdot),\tau^0(\cdot))$ The efficiency of $(x^0(\cdot),(x^0_\gamma(\cdot),y^0(\cdot),y^0_\gamma(\cdot),\Lambda,\mu^0(\cdot),\nu^0(\cdot),\tau^0(\cdot))$ is implied also by the weak duality theorem.

We shall give now a converse duality theorem, by changing some of the hypotheses.

Theorem 6: (Converse Duality) Let $(x^0(\cdot),(x^0_\gamma(\cdot),y^0(\cdot),y^0_\gamma(\cdot),\Lambda,\mu^0(\cdot),\nu^0(\cdot),\tau^0(\cdot))$ be an efficient solution to dual (MZD) and suppose satisfied the following conditions:

(i) $\left(\overline{x}(\cdot),\overline{y}(\cdot)\right)$ is an efficient solution of primal(MP).

(ii) The hypotheses of Theorem 4 hold at $(x^0(\cdot),(x^0_\gamma(\cdot),y^0(\cdot),y^0_\gamma(\cdot),\Lambda,\mu^0(\cdot),\nu^0(\cdot),\tau^0(\cdot))$

The $x^0(\cdot),y^0(\cdot)$ is an efficient solution to (MP). Moreover, we have the equality

$$\pi(x^0(\cdot),y^0(\cdot))=\overline{\delta}(x^0(\cdot),x^0_\gamma(\cdot),y^0(\cdot),y^0_\gamma(\cdot),\Lambda,\mu^0(\cdot),\nu^0(\cdot),\tau^0(\cdot))$$

The proof follows from the weak duality theorem.

4 Conclusions

We introduced a new class of multitime multiobjective variational problems of minimizing a vector of functionals of curvilinear integral type. Based on the normal efficiency conditions for multitime multiobjective variational problems, we studied duals of Mond-Weir type, generalized Mond-Weir-Zalmai type and under some assumptions of (ρ,b)-quasiinvexity, duality theorems are stated. We gave weak duality theorems, proving that the value of the objective function of the primal cannot exceed the value of the dual. Moreover, we studied the connection between values of the objective functions of the primal and dual programs, in direct and converse duality theorems. To the best of our knowledge, the results in §3 are new and they have not been reported in literature.

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